DETECTING GEODESIC CIRCLES IN HYPERBOLIC SURFACES WITH PERSISTENT HOMOLOGY

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Abstract. In this paper we provide conditions under which a geodesic circle on a hyperbolic surface admits arbitrarily small geodesically convex neighborhoods. This implies that persistent homology using selective Rips complexes detects the length and the position of such a loop via persistent homology in dimensions one, two, or three. In particular, if a surface has a unique systole, then the systole can always be detected with persistent homology. The existential results of the paper are complemented the by corresponding quantitative treatments which explain the parameters of selective Rips complexes and conditions under which the detection occurs via the standard Rips complexes. In particular, if a surface has a unique systole, then the parameters depend on the first spectral gap in the length spectrum.

1. Introduction

Persistent homology is a well established tool in theoretical and applied topology. It encodes topological and geometric information when combined with Rips complexes on sufficiently tame metric spaces. While encoding of the homology of an underlying space is well understood as it happens at small scales, the precise nature of geometric information carried by persistent homology is not well understood in most cases. Known results of this nature include one-dimensional persistent homology of geodesic spaces (it encodes the shortest homology base by [16, 17]), parts of persistent homology of ellipses [2] and regular polygons [3], and the complete homotopy type of the Rips filtration of a circle [1].

Especially [1] is of great interest: it demonstrates that the Rips complexes of a circle attain the homotopy types of odd-dimensional spheres and thus a one-dimensional geometric object (a loop) generates higher-dimensional algebraic objects in persistent homology. This idea has led to [18] in which a theory for the detection of parts of the persistent homology of a subset within the persistent homology of the ambient space is presented. In particular, it turns out that under specific conditions a part of the persistent of a loop in a space can be observed in the persistent homology of the space itself. While this part consists of odd-dimensional homology elements in case the underlying loop is a geodesic circle γ (i.e., a circle equipped with a geodesic metric, see Preliminaries below for more details), it turns out that an additional two-dimensional homology element may also be generated from the geometric position of γ when γ is not a member of a shortest homology base. While this two-dimensional element requires somehow generous conditions of the neighborhood of γ, it turns out that a modification of Rips complexes allows us

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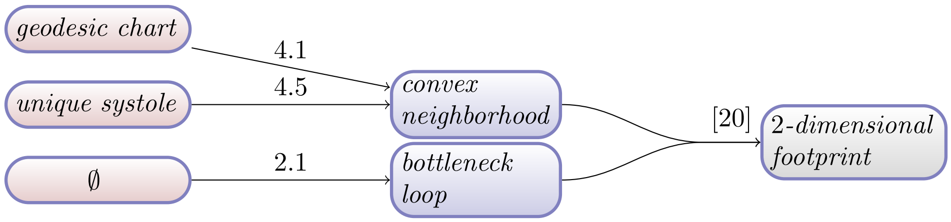

FIGURE 1. A scheme of two-dimensional footprint detection as described by Theorem 6.1.

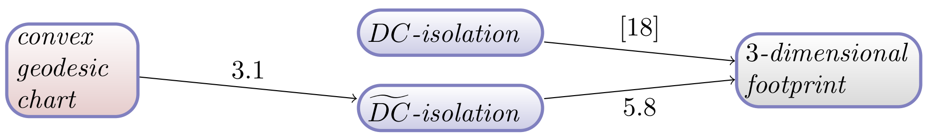

FIGURE 2. A scheme of three-dimensional footprint detection as described by Theorem 6.3.

to deduce its appearance under fairly general conditions. The main result of [20] states that each geodesic circle, which is a locally shortest loop and admits an arbitrarily small geodesically convex neighborhood can be detected either with one- or two-dimensional persistent homology using appropriate selective Rips complexes.

In this paper we consider the results of [18] and [20] in the setting of complete orientable hyperbolic surfaces: we provide simple conditions under which geodesic circles induce a two- or a three-dimensional footprint in persistent homology. It turns out that the somewhat technical conditions of the two mentioned paper can be deduced from the existence of sufficiently large geodesic charts. For a scheme of our results see Figures 1 and 2. The technical results leading to such a connection include the existence of geodesically convex neighborhoods and an introduction of $\widetilde{DC}$ isolated loops. The parameters of the results are also quantified (after Theorem 6.1), leading to specific bounds on parameters of required selective Rips complexes and settings in which the detection takes place with classical Rips complexes. In particular, when the systole of a surface is unique, we can deduce that the parameters of the selective Rips complexes depend on the first spectral gap in the length spectrum.

The paper consists of two distinct parts. The first part is a treatment of geometrical properties in the context of differential geometry. It contains Preliminaries in Section 2, Retractible neighborhoods in Section 3 and the existence of geodesically convex neighborhoods in Section 4. The second part contains the applications to persistent homology. Section 5 contains preliminaries and adaptations of results on persistent homology to our setting. In Section 6 the results of the previous sections are combined to draw the main conclusions of the paper.

2. Preliminaries on differential geometry

Our objects of interest will be orientable smooth hyperbolic (i.e., having potentially non-constant negative Gaussian curvature: $K < 0$) surfaces equipped with a Riemannian metric. We will assume that the surfaces are complete. In particular, the geodesics are defined for all times, and as a consequence, every pair of points

x, y on a surface with $d(x, y) = \ell$ is connected by a **shortest geodesic** of length ℓ. We should point out that the term "geodesics" in this paper refers to locally shortest paths as is common in the context of differential geometry [9, 6], which is prevalent in this paper. In contrast, term "geodesics" or "geodesic segment" in some related works of more general metric context [5, 12, 7, 16, 17, 18, 20] refers to isometric embeddings of intervals. Such paths (and their traces) will be referred to as **shortest geodesic** in this paper. A subset of a surface $\mathcal{S}$ is **geodesically convex** if for each pair of points $x, y \in \mathcal{S}$ each shortest geodesic between them lies in A.

Given $c > 0$ let S^1_c denote a circle equipped with a geodesic metric (meaning the distance between any two points on S^1_c is the length of a shortest segment between the points) of circumference c. A **geodesic circle** on a surface $\mathcal{S}$ is an isometric embedding of S^1_c into $\mathcal{S}$ for some $c > 0$. We will frequently identify a geodesic circle with its trace. Loop $\alpha \subset \mathcal{S}$ is a **bottleneck loop** if there exists a neighborhood of α in which α is the shortest representative of its free homotopy class. The equator on a sphere is a geodesic circle which is not a bottleneck loop. It is not hard to construct a bottleneck loop, which is not a geodesic circle.

2.1. **Variations of Arc Length.** (see [9, p.238 and p.339] for some background)

It is well known that small perturbations of closed geodesics increase the length of the perturbed path in our setting, which implies that geodesic circles are automatically bottleneck loops. While the formal statement of this fact for our purposes could be deduced from Theorem 3.1, we recall the argument for the illustrative purposes.

Let $\gamma(s) : [0, L] \to \mathcal{S}$ be a naturally parametrized smooth loop. A variation is a smooth map

$$h : [0, L] \times (-\varepsilon, \varepsilon) \to \mathcal{S}$$
$$(s, t) \mapsto h(s, t)$$

such that $h(s, 0) = \gamma(s)$ and $h(0, t) = h(L, t)$ (and the coincidence also holds for all derivatives). Denote by

$$V(s) = \frac{\partial h}{\partial t}(s, 0)$$

the *variational vector field.* One can associate h to V by $h(s, t) = \exp_{\gamma(s)}(tV)$. Let

$$\mathcal{L}(t) = \int_0^L \left| \frac{\partial h}{\partial s}(s, t) \right| ds$$

be the length of the variation. Then

$$\mathcal{L}'(0) = -\int_0^L A(s) \cdot V(s)\, ds$$

where $A(s) = \frac{D}{ds}\frac{\partial h}{\partial s}$ (this is zero for a geodesic) and

$$\mathcal{L}''(0) = \int_0^L \left(\left| \frac{D}{ds} V(s) \right|^2 - K(s)\, |V(s)|^2 \right) ds.$$

Here the variational vector field $V(s)$ is orthogonal to $\gamma'(s)$. As each loop (and its length) in $\mathcal{S}$ can be approximated by a smooth loop, we obtain the following result.

Theorem 2.1. *Let $\gamma(s)$ be a simple closed geodesic on $\mathcal{S}$. If the Gaussian curvature satisfies $K(s) < 0$ along the curve then γ is a bottleneck loop.*

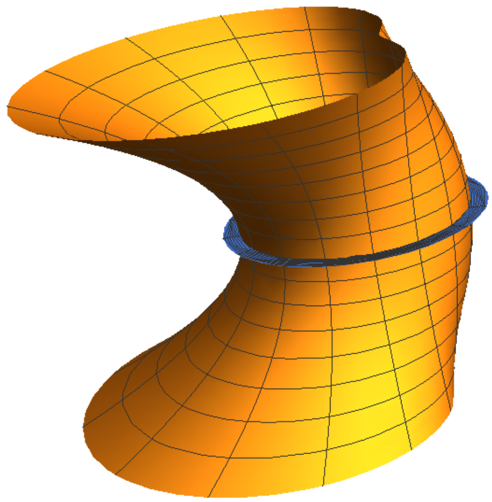

FIGURE 3. Surface has some positive Gauss curvature, but the geodesics is still minimal.

Proof. By the discussion above $\mathcal{L}'(0) = 0$ along γ and there exists $\kappa > 0$ such that $\mathcal{L}''(0) > \kappa$ on some tubular neighborhood of γ. Hence every nontrivial variation strictly increases length. □

Remark. The converse does not hold in general, that is, if every variation increases length it does not necessarily mean that $K(s)$ is negative.

Suppose that $K(s) = \cos s - \frac{1}{2}$ and let $V(s) = f(s)(\gamma'(s) \times \vec{n})$. Then the above integral for the second derivative is

$$\int_0^{2\pi} \left(f'(s)^2 - (\cos s - \frac{1}{2}) f(s)^2 \right) ds\,.$$

Write $f(s) = a_0 + \sum_{n=1}^{\infty}(a_n \cos(ns) + b_n \sin(ns))$ using the Fourier series and the integral equals

$$2\pi a_0(\frac{1}{2}a_0 - a_1) + \sum_{n=1}^{\infty} \pi(n^2 + \frac{1}{2})(a_n^2 + b_n^2) =$$

$$= \pi(a_0 - a_1)^2 + \frac{\pi}{2}a_1^2 + \sum_{n=1}^{\infty} \pi(n^2 + \frac{1}{2})b_n^2 + \sum_{n=2}^{\infty} \pi(n^2 + \frac{1}{2})a_n^2\,.$$

The minimal value (zero) is clearly attained when all $a_i, b_j = 0$, so for every nontrivial variation the integral is strictly larger than zero even though $K(s)$ is not strictly negative.

2.2. **Geodesic coordinates.** (see [15, p.242] for more details on the topic)

Let $\mathcal{S}$ be an orientable surface and let $\gamma(v)$ be a naturally parametrized simple closed geodesic on $\mathcal{S}$. Let $\vec{A}(v)$ be a vector field along γ, perpendicular to γ and

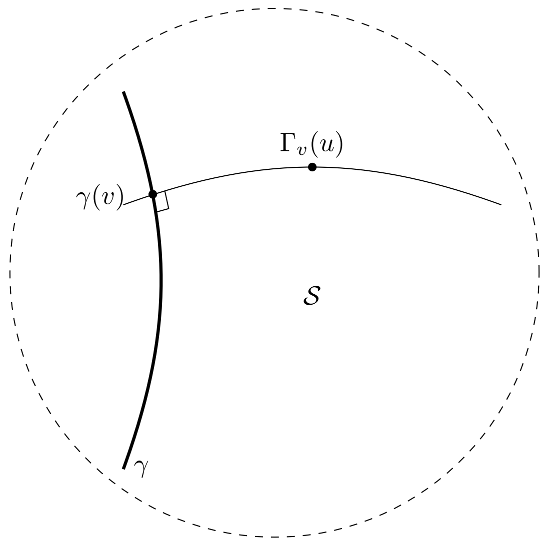

FIGURE 4. Geodesic coordinates.

$|\vec{A}(v)| = 1$. Let $\sigma(u, v) = \Gamma_v(u)$ be the resulting parametrization of the surface, see Figure 2.2.

Theorem 2.2 (See [15, Proposition 9.5.1] for a proof)**.** *The parametrization $\sigma(u, v) = \Gamma_v(u)$ is a chart for $\mathcal{S}$ in a neighbourhood of any point $(0, v)$ with first fundamental form given by*

$$\begin{bmatrix} 1 & 0 \\ 0 & G(u, v) \end{bmatrix}$$

and with $G(0, v) = 1$ and $G_u(0, v) = 0$.

To summarize, the parametrization $\Gamma_v(u)$ is such that $u = const.$ and $v = const.$ form an orthogonal system of curves, with $\Gamma_v(0)$ is a geodesic curve and $\Gamma_{v_0}(u)$ is a geodesic curve and the first fundamental form is given by

$$\begin{bmatrix} 1 & 0 \\ 0 & G(u, v) \end{bmatrix} \quad \text{with} \quad G(0, v) = 1 \quad G_u(0, v) = 0\,.$$

Theorem 2.2 describes geodesic coordinates along γ at any point of a geodesic γ. When γ is a closed geodesic of length ℓ on an orientable surface, we can combine these into a consistent **geodesic coordinates** along γ. In particular, there is a width $\varepsilon > 0$ and a smooth embedding

$$H : (-\varepsilon, \varepsilon) \times [0, \ell]/_{0 \simeq \ell} \to \mathcal{S}$$
$$H : (u, v) \mapsto H(u, v)\,,$$

such that $H(u, v) = \Gamma_v(u)$. Map H will represent such geodesic coordinates throughout the paper and will also be referred to as **geodesic chart**.

3. Retractable neighborhood

The main goal of this section is to formalize the following phenomenon: given a path α in a geodesic chart around a closed geodesic γ, sliding α "perpendicularly" towards γ results in ever shorter paths, see Figure 5. This is done by Theorem 3.1:

(1) First for paths α "parallel" (in terms of geodesic coordinates) to γ.
(2) Then for general paths α.
(3) Finally for sliding α towards a parallel to γ.

Theorem 3.1. *Let $\mathcal{S}$ be an orientable surface, let $\gamma(v)$ be a naturally parameterized simple closed geodesic of length ℓ on $\mathcal{S}$, and assume the Gaussian curvature K of $\mathcal{S}$ is negative on γ. Choose $\varepsilon > 0$ such that the Gaussian curvature is negative on the ε-geodesic chart of γ*

$$H : (-\varepsilon, \varepsilon) \times [0, \ell]/_{0\simeq\ell} \to \mathcal{S}$$
$$H : (u, v) \mapsto H(u, v)$$

Then the following conclusions hold:

(1) *For any $a < b \in [0, \ell]$ the length of the curve $H(u_1, v)$, $v \in [a, b]$ is strictly smaller then the length of the curve $H(u_2, v)$, $v \in [a, b]$ for $|u_1| < |u_2|$.*
(2) *Denote by $\kappa_t(u, v) = H((1-t)u, v)$ the deformation retraction of the (image of the) geodesic chart onto γ. Let $H(u(\tau), v(\tau))$, $\tau \in [a, b]$ be a curve on $\mathcal{S}$. Then $\frac{d}{dt}\kappa_t(u(\tau), v(\tau)) < 0$, i.e. the length of the curve $\kappa_t(u(\tau), v(\tau))$ is decreasing as t increases from 0 to 1.*
(3) *Choose $0 < \delta < \varepsilon$. Let $H(u(\tau), v(\tau))$, $\tau \in [a, b]$ be a curve on $\mathcal{S}$ with $u(\tau) \geq \delta$. Define $\nu_t(u, v) = H\big((1-t)u(\tau) + t\delta, v(\tau)\big)$. Then $\frac{d}{dt}\nu_t(u(\tau), v(\tau)) < 0$, i.e. the length of the curve $\nu_t(u(\tau), v(\tau))$ is decreasing as t increases from 0 to 1.*

Proof. (1) In geodesic coordinates (see Theorem 2.2 for the properties that will be used throughout this argument) the Gaussian curvature is expressed as

$$K = -\frac{1}{2\sqrt{G}}\left(\frac{G_u}{\sqrt{G}}\right)_u = -\frac{1}{2\sqrt{G}}\left(\frac{G_{uu}\sqrt{G} - \frac{G_u^2}{2\sqrt{G}}}{\sqrt{G}}\right),$$

by the Brioschi formula. This implies that $G_{uu}(0, v) > 0$. Since $G_u(0, v) = 0$ by the construction of geodesic coordinates, we get that $G_u(u, v) > 0$ for $u > 0$ and $G_u(u, v) < 0$ for $u < 0$. In particular $G(u, v) > 1$ for $|u| > 0$. The curve $H(u_0, v)$, $v \in [a, b]$ has length equal to

$$L(u_0) = \int_a^b \sqrt{G(u_0, v)}\, dv\,.$$

Since $L'(u_0) = \int_a^b \frac{G_u(u_0,v)}{2\sqrt{G(u_0,v)}}\, dv$, $L(u)$ is decreasing for negative u and increasing for positive u and this means that the length of the curve $H(u_1, v)$, $v \in [a, b]$ is strictly smaller then the length of the curve $H(u_2, v)$, $v \in [a, b]$ for $0 < u_1 < u_2$ (and similarly for $u_2 > u_1 > 0$).

(2) The length of the curve is

$$L(t) = \int_a^b \sqrt{(1-t)^2 u'(\tau)^2 + G((1-t)u(\tau), v(\tau))v'(\tau)^2}\, d\tau\,.$$

The derivative

$$\frac{dL(t)}{dt} = \int_a^b \frac{-2(1-t)u'(\tau)^2 + G_u\Big((1-t)u(\tau), v(\tau)\Big)v'(\tau)^2\big(-u(\tau)\big)}{2\sqrt{(1-t)^2 u'(\tau)^2 + G\big((1-t)u(\tau), v(\tau)\big)}}\, d\tau$$

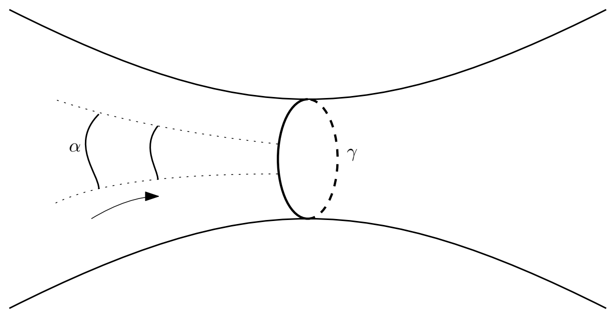

FIGURE 5. Sliding a path α towards γ results in ever shorter paths.

is always negative, since $-2(1-t)u'(\tau)^2$ is clearly negative and G_u is positive for positive u and negative for negative u. So as t flows from $0 \to 1$, $L(t)$ decreases. The proposed deformation retraction $\kappa_t(u,v) = H((1-t)u, v)$ indeed shortens all paths (Figure 5).

(3) The proof is mostly the same as that of (2). □

Remark: The theorem above deals with the case where K is negative. However, a version of the theorem for non-negative K can be proved in the same manner: we could replace *negative curvature* by *non-negative curvature* and change all strict inequalities (except for $\varepsilon > 0$, of course) to non-strict inequalities.

4. Geodesic convexity of neighborhoods

In this section we consider the existence of small geodesically convex neighborhoods of geodesic circles. We will first discuss why some of the assumptions are necessary for such a statement.

If there is no condition on the Gaussian curvature then the geodesic chart H (in particular, its image) is not geodesically convex in general. For example, think about the equator on a sphere, which has no small geodesically convex neighborhood. Even if $K \leq 0$ everywhere a geodesic chart is not necessarily geodesically convex at all widths ε. The surface in Figure 6 has $K \leq 0$ everywhere. There is a (large enough) cylindrical subset of a geodesic circle circumventing any of the visible holes, that is not geodesically convex. Furthermore, not every simple closed geodesic is geodesically convex as Figure 7 shows, and the same goes for its small neighborhoods.

Now that the required conditions are established, we proceed by proving the existence of geodesically convex neighborhoods provided a wide enough geodesic chart exists.

Theorem 4.1. *Let $\mathcal{S}$ be a complete orientable surface with $K < 0$ and let γ be a naturally parametrized simple closed geodesics of length ℓ. Suppose that*

$$H\left((-D,D) \times [0,\ell]/_{0\simeq\ell}\right) \subset \mathcal{S}$$

is a geodesic chart around γ, where $D > \ell/4$. Then for each $\delta \leq \frac{D-\frac{\ell}{4}}{2}$, neighborhood $H\left((-\delta,\delta) \times [0,\ell]\right)$ is geodesically closed. Furthermore, for each $\delta' < \frac{D-\frac{\ell}{4}}{2}$, neighborhood $H\left([-\delta',\delta'] \times [0,\ell]\right)$ is geodesically closed.

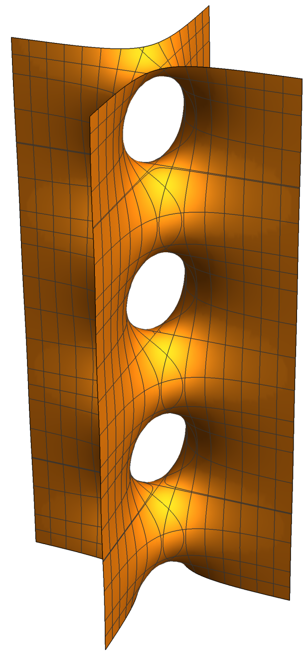

FIGURE 6. Scherk's singly periodic surface, also called the Scherk-tower.

Proof. The situation is depicted in Figure 8. Choose $p, q \in H((-\delta,\delta)\times[0,\ell]))$ and let α be a shortest geodesic between them. Assume α does not lie entirely in $H((-\delta,\delta)\times[0,\ell]))$. We analyse the situation by considering two separate cases:

(1) Assume α lies in $H((-D,D)\times[0,\ell])$. Using case (3) of Theorem 3.1 we can slide the part of α lying outside $H((-\delta,\delta)\times[0,\ell]))$ inside $H((-\delta,\delta)\times[0,\ell]))$ while keeping endpoints p, q intact and decreasing the length of the path. The result is a path between p and q which is shorter than α, a contradiction.

(2) Assume α does not lie in $H((-D,D)\times[0,\ell])$, see Figure 9. On one hand this means α starts at p, has to reach the complement of $H((-D,D)\times[0,\ell])$ (which is at distance at least $\ell/4+\delta$ from p) and then has to reach q again (meaning it has to traverse a distance at least $\ell/4+\delta$ from p again), resulting in a lower bound $\ell/2+2\delta$ for the length of α.

    On the other hand we can construct a different path from p to q using γ. Start at p and follow a geodesic towards γ (of length less than δ) to reach $P_0 \in \gamma$. In a similar fashion initiate a new path segment by starting at at q and follow a geodesic towards γ (of length less than δ) to reach $Q_0 \in \gamma$. Connect P_0 and Q_0 by a path along γ of length at most $\ell/2$. Concatenating these three paths we obtain a path from p to q of length less than $\ell/2+2\delta$. This contradicts the lower bound of the previous paragraph.

We conclude that α lies entirely in $H((-\delta,\delta)\times[0,\ell]))$.

The statement for closed neighborhoods can be proved in the same way. $\square$

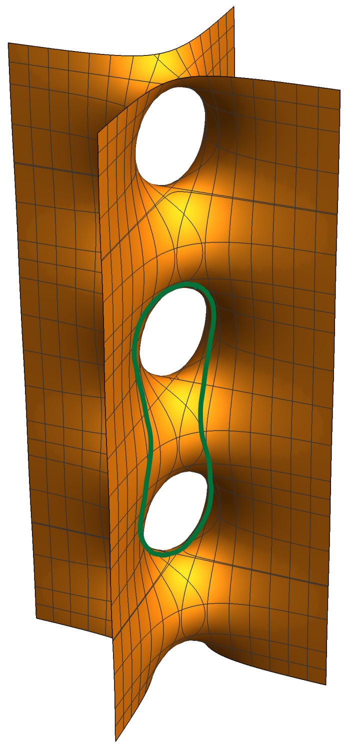

FIGURE 7. Closed geodesic which is not geodesically closed and hence not a geodesic circle.

Remark 4.2. Theorem 4.1 is stated for the case $K < 0$ as this is one of the assumptions of our eventual applications to persistent homology, and the proof of case (1) follows fairly easily. However, Theorem 4.1 as stated also holds when $K \leq 0$. In order to prove it we only need to modify case (1) of the corresponding proof as follows.

Assume $\alpha = H(u(t), v(t)) \colon [0,1] \to \mathcal{S}$ lies in $H\left((-D, D) \times [0, \ell]\right)$. Without the loss of generality we may assume that $p = H(\delta, v_p) = H(u(0), v(0)), q = H(\delta, v_q), H(u(1), v(1))$ for some $\delta < D$, and $u(t) > \delta, \forall t \in (0,1)$. The length of α equals

$$L_1 = \int_{t=0}^{1} \sqrt{u'^2 + G(u(t), v(t)) \cdot v'^2} dt.$$

By the same reasoning the length of the projection of α onto $H([-\delta, \delta] \times [0, \ell])$ equals

$$L_2 = \int_{t=0}^{1} \sqrt{G(\delta, v(t)) \cdot v'^2} dt.$$

By the assumptions:

(1) u' is non-zero at some interval as $u(t) > \delta, \forall t \in (0,1)$, and

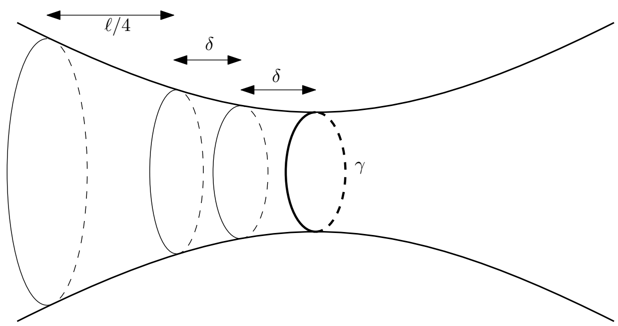

FIGURE 8. Setup of Theorem 4.1.

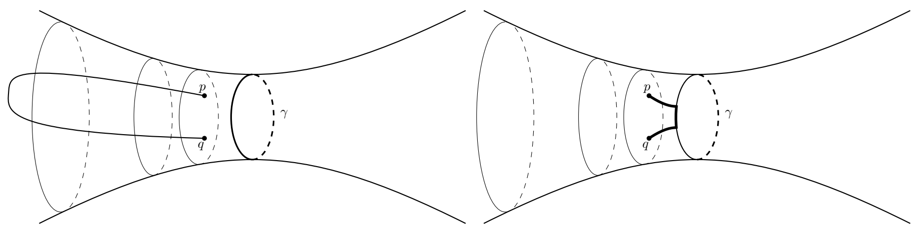

FIGURE 9. A sketch of a part of the proof of Theorem 4.1. If a shortest geodesic between p and q reaches the complement of $H\left((-D, D) \times [0, \ell]\right)$ as depicted on the left, then there exists a shorter path between the two points (on right) along γ.

(2) $G(u(t), v(t)) \geq G(\delta, v(t))$ as $G(0, v) = 1$ and $G_u(u, v) > 0$ for $u > 0$ (see part (1) of the proof of Theorem 3.1).

As a consequence, $L_1 > L_2$.

The conditions of Theorem 4.1 contain the size of a geodesic chart. Throughout the rest of this section we explain how this condition may be dropped in the case of unique shortest closed geodesic on a surface.

We first recall the Cartan-Hadamard Theorem in Riemannian geometry.

Theorem 4.3. *Let $\mathcal{S}$ be a complete surface with $K \leq 0$ and $p \in K$. Then the map*

$$\exp_p : T_p\mathcal{S} \to \mathcal{S} \qquad p \in \mathcal{S}$$

is a universal covering projection.

Theorem 4.4. *Let $\mathcal{S}$ be a complete orientable surface with $K \leq 0$ and let γ be a simple contractible loop. Then γ bounds a simple region (diffeomorphic to disc) in $\mathcal{S}$.*

Proof. Let γ be a simple contractible loop. Then the lift to the universal covering space is also a simple loop $\tilde{\gamma}$. By the Jordan-Schönflies Theorem $\tilde{\gamma}$ bounds a simple region $\mathcal{D}$.

We now intend to prove that the covering projection restricted to $\mathcal{D}$ is injective. For each non-trivial deck transformation g on the universal cover we have $\tilde{\gamma} \cap g(\tilde{\gamma}) = \emptyset$ as γ is simple. Furthermore, if $g(\mathcal{D}) \subseteq \mathcal{D}$, then g would have a fixed point (any point of the $\bigcap_{n\in\mathbb{N}} g^n(\mathcal{D}) \neq \emptyset$ is a fixed point by a standard argument), a contradiction. Hence $\mathcal{D} \cap g(\mathcal{D}) = \emptyset$ and thus the covering projection is injective on $\mathcal{D}$. Consequently, the image of $\mathcal{D}$ by the covering projection is a homeomorphic image of $\mathcal{D}$, whose boundary (in $\mathcal{S}$) is γ. $\square$

The (unmarked) **length spectrum** of a manifold is the collection of lengths of all closed geodesics. Each compact complete hyperbolic manifold has a discrete length spectrum. For a proof see, for example, [4, Lemma 3.1 and Remark 3.2]. Each shortest closed geodesic in a manifold is called **systole**. Each systole is a geodesic circle. The length of a systole is hence the first value of the length spectrum.

Proposition 4.5 combined with Remark 4.6 states that if the systole of a hyperbolic surface is unique, a choice of D as in Theorem 4.1 can be made.

Proposition 4.5. *Let $\mathcal{S}$ be a compact complete orientable surface with $K \leq 0$. Assume a closed loop γ of length ℓ is the unique systole of $\mathcal{S}$ and $L > \ell$ is the second value of the length spectrum, i.e., each closed loop of length less than L is either contractible or homotopic to γ.*

Then parameter D of Theorem 4.1 can be chosen to be $D = L/2 - \ell/4 > L/4$, i.e.,

$$H\left((-D, D) \times [0, \ell]/_{0 \simeq \ell}\right) \subset \mathcal{S}$$

is a geodesic chart around γ.

Remark 4.6. Before we embark on the proof of Proposition 4.5 we briefly explain why a choice of L in as assumed in Proposition 4.5 always exists if γ. As $\mathcal{S}$ compact then, as was mentioned before Proposition 4.5, the length spectrum of $\mathcal{S}$ is discrete and hence L can be chosen to be the second value of the length spectrum (and use the fact that each non-contractible loop has a representative as a closed geodesic). In particular, the pair ℓ, L represents the first spectral gap of the length spectrum.

Proof of Proposition 4.5. We need to prove that H is injective on $(-D, D) \times [0, \ell]$. Assume on the contrary, that there exists $D' < D$ such that there exist two geodesics which:

- start at points $p, q \in \gamma$;
- are perpendicular to γ at these points;
- have their first point of intersection at z with $d(z, \gamma) = D'$.

See Figure 10 for a sketch one such situation. The corresponding geodesic segments (from z to p and q, and a shortest segment from p to q along γ) form a geodesic triangle T with angles being $\pi/2, \pi/2$ and a non-trivial angle. Triangle T as a loop is of length at most $D + D + \ell/2 < L$ hence T is contractible. By Theorem 4.4 T bounds a disc. By the Gauss-Bonnet Theorem we get that $\iint_D K dS > 0$, a contradiction. $\square$

5. Preliminaries on persistent homology and footprint detection

Persistent homology is a type of parameterized version of homology. Ever since its introduction two decades ago the corresponding theory and applications witnessed intense development that expanded onto other fields of mathematics and science. For a general exposition on the topic see [10]. In this paper we will focus on a specific setting of hyperbolic surfaces. On the other hand, our treatment will be slightly more general than the standard approaches in persistent homology as we will not restrict our choice of coefficients to fields. We proceed by briefly presenting our setting. For a similar setting see [16, 17, 18, 20].

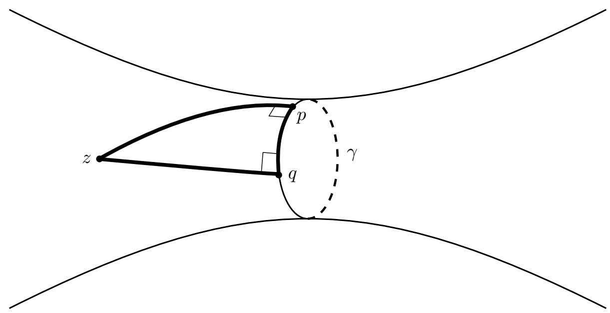

FIGURE 10. A sketch of a part of the proof of Proposition 4.5: two geodesics perpendicular to γ having z as the first point of their intersection. Note that geodesics could also emerge from γ in different directions.

Let X be a metric space and fix a scale $r > 0$. The **Rips complex** $\mathrm{Rips}(X; r)$ is an abstract simplicial complex with the vertex set X defined by the following rule: a finite $\sigma \subset X$ is a simplex iff $\mathrm{Diam}(\sigma) < r$.

Definition 5.1. [20] Let Y be a metric space, $r_1 \leq r_2, n \in \mathbb{N}$. **Selective Rips complex** $\mathrm{sRips}(Y; r_1, n, r_2)$ is an abstract simplicial complex defined by the following rule: a finite subset $\sigma \subset Y$ is a simplex iff the following two conditions hold:

(1) $\mathrm{Diam}(\sigma) < r_1$;
(2) there exist subsets $U_0, U_1, \ldots, U_n \subset U$ of diameter less than r_2 such that $\sigma \subset U_0 \cup U_1 \cup \ldots \cup U_n$.

The collection of Rips complexes of X for all positive $r > 0$ can be assembled together into the Rips filtration $\{\mathrm{Rips}(X; r)\}_{r>0}$ of X consisting additionally of bonding maps $i_{r_1,r_2}\colon \mathrm{Rips}(X; r_1) \to \mathrm{Rips}(X; r_2)$, which are natural inclusions (identities on vertices) for all $r_1 < r_2$. Obtaining a filtration of selective Rips complexes we are required to make a more specific choice of positive increasing functions $r_1(t) \leq r_2(t)$.

Persistent homology is obtained by applying the homology functor to any filtration. When coefficients of a homology form a field the resulting persistent homology may under appropriate conditions (for example, if X is finite) be presented by a collection of intervals, which give rise to two know visualizations of persistent homology: persistence diagram and barcodes (see [10] for details).

5.1. **Geodesic circles and persistent homology.** We next present the known results which explain how geodesic circles in Riemannian manifolds generate algebraic objects (**footprints**) in persistent homology in various dimensions. Let γ be a geodesic circle of length ℓ in a Riemannian surface X and fix coefficients G for all homology groups in this section. Theorem 5.2 states that if γ is a member of a shortest homology base, then it induces a one-dimensional footprint which dies at $|\gamma|/3$. In this case it is the topological nature of γ that induces the footprint and hence the name **topological footprint**.

Theorem 5.2. [16] *[Generating 1-dimensional (topological) footprint] Let X be a compact semi-locally simply connected geodesic space and let G be an Abelian group. If γ is a member of a shortest homology base, then for each $r > 0$ a discretization* *of γ (called r-sample of γ in* [16]*) at scale r represents a homology class $Q_r \in H_1(\mathrm{Rips}(X;r);G)$ such that:*

- *for each $r_1 < r_2$ the inclusion $\mathrm{Rips}(X;r_1) \to \mathrm{Rips}(X;r_2)$ induced map maps Q_{r_1} to Q_{r_2}, and*
- *$Q_r \neq 0$ iff $r < |\gamma|/3$.*

However, the focus of this paper is to also detect geodesic circles which are not contained in any shortest homology base and to detect γ via higher-dimensional homology.

In [18] a generic method for detecting γ through higher dimensional persistent homology is described. The result is inspired by [1], in which authors demonstrate that the homotopy type of a circle via Rips complexes attains all even dimensional spheres. It turns out that this result may be used to prove that $\gamma \subset X$ sometimes induces even-dimensional homology elements in persistent homology of X via Rips filtration. Additionally, two-dimensional elements may also appear when γ is not a member of a shortest homology base. It turns out that these two-dimensional elements appear under very weak assumptions if selective Rips complexes are used [20].

We continue by providing technical prerequisites, details and adjustments of these two results as we will later combine them with the results of the previous sections.

5.2. **Detection via selective Rips complexes.** We start with detection results via selective Rips complexes as described in [20]. Despite being chronologically more recent than the approach via deformation contractions described in the following subsection, we describe this setting first due to its simplicity. Broadly speaking, it turns out that each γ which is a geodesic circle, a bottleneck loop and has arbitrarily small convex neighborhood can be detected either with 1-dimensional persistent homology (in case γ is a member of a shortest homology base by [16]) or by a 2-dimensional persistent homology via selective Rips complexes. In this case it is the geometric property of a neighborhood of γ that induces the footprint in the absence of a topological footprint and hence the name **geometric footprint**.

Theorem 5.3. [20] *[Generating 2-dimensional (geometric) footprint] Let X be a geodesic locally compact, semi-locally simply connected space and let G be an Abelian group. Assume α is a geodesic circle in X satisfying the following properties:*

1. *α is a bottleneck loop;*
2. *α is homologous to a non-trivial G-combination of loops $\beta_1, \beta_2, \dots, \beta_k$ of length at most $|a|$, none of which is homotopic to α or α^-;*
3. *α has arbitrarily small geodesically convex neighborhood.*

Then there exist bounds $B_1 > |\alpha|/3$ and $B_2 > 0$ such that for all increasing bijections $a \geq b\colon (0,\infty) \to (0,\infty)$, and for all $r > 0$ such that $B_1 > a(r) > |\alpha|/3$ and $B_2 > b(r)$, there exists a non-trivial

$$Q_r \in H_2(\mathrm{sRips}(X; a(r), 2, b(r)); G)$$

satisfying the following properties:

1. *$\forall r_1 < r_2$ with $a(r_i) > |\alpha|/3$ and $b(r_i) < B, \forall i$, we have $i^G_{r_1,r_2}(Q_{r_1}) = Q_{r_2}$, where $i^G_{r_1,r_2}\colon H_2(\mathrm{sRips}(X; a(r_1), 2, b(r_1)); G) \to H_2(\mathrm{sRips}(X; a(r_2), 2, b(r_2)); G)$ is the natural inclusion induced map.*
2. *$\forall q : a(q) \leq |\alpha|/3$ there exists no Q_q with $i_{q,r}(Q_q) = Q_r$.*

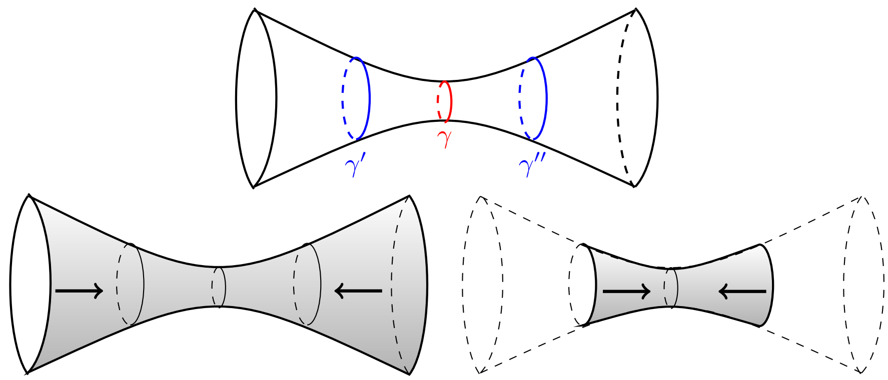

FIGURE 11. A $\widetilde{DC}$-isolated loop γ.

5.3. **Detection via Rips complexes.** The second way of detection was described in [18] and works in a fairly general setting. For our purposes we will adapt the results of [18] to our setting. We start by defining deformation contractions, i.e., maps inducing homotopy equivalence on Rips complexes (observe relation with retractable neighborhoods in Section 3, it will be employed later).

Definition 5.4. [13, 18] Let X be a metric space and $A \subset X$. A continuous map $F\colon X \times [0,1] \to X$ is called a **deformation contraction** (we will abbreviate it as DC and write $X \xrightarrow{DC} A$) if:

(1) $F(x,0) = x, F(x,1) \in A, F(a,t) = a, \forall x \in X, a \in A, t \in [0,1]$, and
(2) $d(F(x,t'), F(y,t')) \leq d(F(x,t), F(y,t)), \forall x, y \in X, t' > t$.

If additionally $d(F(x,t'), F(y,t')) < d(F(x,t), F(y,t))$ holds for all pairs $(x,y) \in (X \setminus A) \times X$ and for all $t' > t$, then F is called a strict deformation contraction (SDC or $X \xrightarrow{SDC} A$).

Proposition 5.5. [18] *Suppose $X \xrightarrow{DC} A$ via a map F. Then the inclusions* $\mathrm{Rips}(A;r) \hookrightarrow \mathrm{Rips}(X;r)$ *are homotopy equivalences for each $r > 0$.*

A local property used in [18] to deduce detection of a loop is that of DC-isolation (deformation contraction isolated). For our purposes a minor modification of this property $\widetilde{DC}$ will be more useful.

Definition 5.6. Suppose $0 < D_1 \leq D_2$. A loop $\gamma \subset \mathcal{S}$ in a complete surface with $K \leq 0$ is $\widetilde{DC}(D_1, D_2)$-isolated if the following conditions hold for closed geodesic neighborhoods $N_1 = \overline{N}(\gamma, D_1/2)$ and $N_2 = \overline{N}(N_1, D_2)$ of γ:

(1) N_2 is geodesically convex.
(2) Sliding along geodesics perpendicular to γ towards γ in a geodesic chart N_2 (see Figure 5 and (2) of Theorem 3.1) induces $N_2 \setminus \mathrm{Int}(N_1) \xrightarrow{SDC} \partial N_1$ and $N_1 \xrightarrow{SDC} \gamma$, see Figure 11.

Note that if γ is $\widetilde{DC}(D_1, D_2)$-isolated then $\overline{N}(\gamma, r)$ is geodesically convex for each $r \leq D_2$ by Theorem 3.1.

Proposition 5.7. *Let $0 < D_1 \leq D_2$ and suppose a loop $\gamma \subset \mathcal{S}$ in a complete surface with $K \leq 0$ is $\widetilde{DC}(D_1, D_2)$-isolated. In this case the boundary ∂N_1 consists of two*

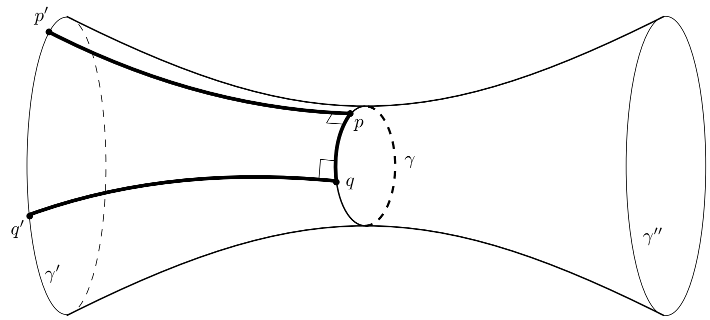

FIGURE 12. A sketch the setting of Proposition 5.7.

simple closed loops denoted by γ', γ''*, see Figure 11. For each* $p \in \gamma$ *let* p' *and* p'' *denote the points on* γ' *and* γ'' *respectively, which are closest to* p*, see Figure 12. Then the following two statements hold:*

(1) *There exists* $D' > |\gamma|/3$ *such that for each* $p, q \in \gamma$ *with* $d(p,q) \geq |\gamma|/3$ *we have* $d(p',q') > |\gamma|/3 + D'$ *and* $d(p'',q'') > |\gamma|/3 + D'$*.*
(2) *For each* $r \in (|\gamma|/3, D')$ *the inclusion induced maps* $\mathrm{Rips}(\gamma'; r) \hookrightarrow \mathrm{Rips}(N_1; r)$ *and* $\mathrm{Rips}(\gamma''; r) \hookrightarrow \mathrm{Rips}(N_1; r)$ *are homotopically trivial.*

Proof. (1) Map $(p,q) \mapsto d(p',q')$ is continuous and (by Definition 5.6) does not attain value $|\gamma|/3$ on a compact domain $\{(p,q) \in \gamma; d(p,q) \geq |\gamma|/3\}$ hence it has a lower bound above $|\gamma|/3$. The same holds for γ'' and the minimum of these two lower bounds is D'.

(2) By Proposition 5.5 we have $\mathrm{Rips}(N_1; r) \simeq \mathrm{Rips}(\gamma; r)$ with the homotopy equivalence arising from our setting (see Definition 5.6) mapping $p' \mapsto p$. Thus $\mathrm{Rips}(\gamma'; r) \hookrightarrow \mathrm{Rips}(N_1; r)$ is homotopic to a map $\mathrm{Rips}(\gamma'; r) \to \mathrm{Rips}(\gamma; r)$ mapping $p' \mapsto p$. By (1) the image of this map is actually contained in $\mathrm{Rips}(\gamma; |\gamma|/3)$, which is homotopy equivalent to the circle by [1]. On the other hand, $\mathrm{Rips}(\gamma; r)$ is homotopy equivalent to a sphere of dimension at least three by [1], hence the map in question is nullhomotopic. $\square$

The property of being DC-isolated was used in [18] to deduce an appearance of odd-dimensional homology elements. In a similar fashion we can prove a similar result for $\widetilde{DC}$ isolated loops in our setting. In this case the combinatorics of Rips complexes from [1] induces the footprint and hence the name **combinatorial footprint**.

Theorem 5.8. *[Generating* 3*-dimensional (combinatorial) footprint via Rips complexes] Suppose* $\mathcal{S}$ *is a complete orientable surface with* $K \leq 0$*,* γ *is a* $\widetilde{DC}(D,D)$ *isolated geodesic circle for some* $D \in (|\gamma|/3, 2|\gamma|/5)$*, and* G *is a group. Then there exists* $D' \in (|\gamma|/3, 2|\gamma|/5)$ *such that the inclusion* $\gamma \hookrightarrow X$ *induces an inclusion*

$$\{G\}_{r \in (|\gamma|/3, D')} \cong \{H_3(\mathrm{Rips}(\gamma; r); G)\}_{r \in (|\gamma|/3, D')} \hookrightarrow \{H_3(\mathrm{Rips}(X; r); G)\}_{r \in (|\gamma|/3, D')}.$$

In particular, for each $r \in (|\gamma|/3, D')$ *there exists a non-trivial*

$$Q_r \in H_3(\mathrm{Rips}(X; r); G)$$

such that:

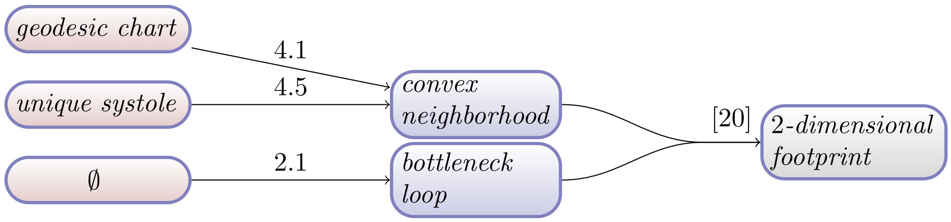

FIGURE 13. A scheme of a geometric footprint detection.

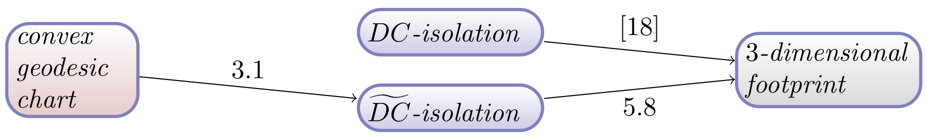

FIGURE 14. A scheme of combinatorial footprint detection.

(1) *$\forall r_1 < r_2$ from $(|\gamma|/3, D')$ we have $i^G_{r_1,r_2}(Q_{r_1}) = Q_{r_2}$, where*

$$i^G_{r_1,r_2} \colon H_3(\mathrm{Rips}(X; r_1); G) \to H_3(\mathrm{Rips}(X; r_2); G)$$

is the natural inclusion induced map.

(2) *$\forall q : a(q) \le |\gamma|/3$ there exists no Q_q with $i_{q,r}(Q_q) = Q_r$.*

Proof. Note that $\{G\}_{r\in(|\gamma|/3,D)} \cong \{H_3(\mathrm{Rips}(\gamma; r); G)\}_{r\in(|\gamma|/3,D)}$ follows from [1] as $\mathrm{Rips}(\gamma; r) \simeq S^3, \forall r \in (|\gamma|/3, D)$. Choose D' as in Proposition 5.7.

We set a Mayer-Vietoris long exact sequence using the notation of Definition 5.6. For a fixed $r \in (|\gamma|/3, D)$ define

$$A = \mathrm{Rips}(N_2; r) \simeq \mathrm{Rips}(\gamma; r),$$

$$B = \mathrm{Rips}(\mathcal{S} \setminus \mathrm{Int}(N_1); r).$$

Note that

$$A \cap B = \mathrm{Rips}(N_2 \setminus \mathrm{Int}(N_1); r) \simeq \mathrm{Rips}(\gamma'; r) \sqcup \mathrm{Rips}(\gamma''; r)$$

as γ' and γ'' are at distance more than r, and $A \cup B = \mathrm{Rips}(\mathcal{S}; r)$ as each subset of $\mathcal{S}$ of diameter less than r is contained either in N_2 or $\mathcal{S} \setminus \mathrm{Int}(N_1)$. From the Mayer-Vietoris sequence we extract the following exact subsequence:

$$H_3(A \cap B; G) \to H_3(A; G) \oplus H_3(B; G) \to H_3(\mathrm{Rips}(X, r); G).$$

By (2) of Proposition 5.6 the first map is trivial, which implies that the second map is an inclusion on $H_3(A; G) = H_3(\mathrm{Rips}(\gamma; r); G)$. The formal conclusion follows from the functoriality of the Mayer-Vietoris sequence. □

6. FINAL RESULTS

In this section we combine the geometric results of initial sections with the footprint detection results of Section 5 to describe detection (footprints) of geodesic circles on hyperbolic surfaces.

6.1. **Results with selective Rips complexes [two-dimensional footprint].** A scheme of the following result is provided in Figure 13.

Theorem 6.1. *Let $\mathcal{S}$ be a compact complete orientable surface with $K < 0$ and let $\gamma \subset \mathcal{S}$ be a geodesic circle. Assume any of the following holds:*

(1) *$H((-D, D) \times [0, \ell]) \subset \mathcal{S}$ is a geodesic chart around γ, where $D > |\gamma|/4$.*
(2) *Loop γ is the unique systole of $\mathcal{S}$.*

Then at $r = |\gamma|/3$ loop γ induces

(i): *a one-dimensional footprint in the sense of Theorem 5.2, or*
(ii): *a two-dimensional footprint in the following sense: there exists a filtration of selective Rips complexes through which γ induces a two-dimensional footprint in the sense of Theorem 5.3.*

Remark 6.2. Loops γ may induce both a one- and a two-dimensional footprint in the sense of Theorem 6.1 if it appears in a shortest homology base of $H_1(\mathcal{S}; G)$ and is homologous to a non-trivial G-combination of loops $\beta_1, \beta_2, \dots, \beta_k$ of length at most $|\gamma|$, none of which is homotopic to γ or γ^-. In this case γ is not a member of each shortest homology base of $H_1(\mathcal{S}; G)$.

Proof. If γ is a member of a shortest homology base, conclusion **(i)** follows by Theorem 5.2. If that is not the case, **(ii)** and the conclusion of Theorem 5.3 follow if γ is a bottleneck loop (which holds always by Theorem 2.1) and has arbitrarily small geodesically convex neighborhood. The existence of the later follows either from Theorem 4.1 for assumption (1), or from Proposition 4.5 for assumption (2). □

Quantification of Theorem 6.1: Concerning conclusion **(ii)** we discuss the parameters of selective Rips complexes and conditions under which the induced two-dimensional footprint can be detected with the usual Rips complexes.

Starting with assumption (1) we have a geodesic chart of width D. Theorem 4.1 implies the existence of a geodesically convex geodesic chart of width $\widetilde{T} = D/2 - |\gamma|/8$. The quantitative analysis of Theorem 5.3 in [20] implies that a selective Rips complex satisfies conclusion **(ii)** if the following holds: for $r > 0$ at which $a(r) = |\gamma|/3$ we need to have $b(r) < \widetilde{T} = D/2 - |\gamma|/8$ (recall notation of Theorem 5.3). In particular, the detection occurs with Rips complexes when $\widetilde{T} > |\gamma|/3$, i.e., $D > 11|\gamma|/12$.

Continuing with assumption (2) we see that $D = L/2 - |\gamma|/4$, where $|g|$ and L are the first two values of the length spectrum of $\mathcal{S}$ and thus $L - |\gamma|$ is the first spectral gap. Consequently, a selective Rips complex satisfies conclusion **(ii)** if the following holds: for $r > 0$ at which $a(r) = |\gamma|/3$ we need to have $b(r) < (L - |\gamma|)/4$, i.e., the upper bound is the quarter of the first spectral gap. In particular, the detection occurs with Rips complexes when $(L - |\gamma|)/4 > |\gamma|/3$, i.e., $L > 7|\gamma|/3$. The ability of Rips complexes to detect γ with a two-dimensional footprint hence depends on the first spectral gap.

6.2. **Results with Rips complexes [three-dimensional footprint].** A scheme of the following result is provided in Figure 14.

Theorem 6.3. *Let $\mathcal{S}$ be a compact complete orientable surface with $K < 0$ and let $\gamma \subset \mathcal{S}$ be a geodesic circle. Assume any of the following holds:*

(1) $H\left([-\widetilde{T},\widetilde{T}]\times[0,\ell]\right)\subset\mathcal{S}$ *is a geodesically convex geodesic chart around* γ*, where* $\widetilde{T}>3|\gamma|/2$.
(2) $H((-D,D)\times[0,\ell])\subset\mathcal{S}$ *is a geodesic chart around* γ*, where* $D>13|\gamma|/4$.
(3) *Loop* γ *is the unique systole of* $\mathcal{S}$ *with* $L>7|\gamma|$ *being the second smallest value of the length spectrum.*

Then at $r=|\gamma|/3$ *loop* γ *induces a three-dimensional footprint in the sense of Theorem 5.8.*

Proof. Assume (1) holds. Then by Theorem 3.1 there exists $\widetilde{D}\in(|\gamma|/3,2|\gamma|/5)$ such that γ is $\widetilde{DC}(\widetilde{D},\widetilde{D})$ isolated, thus the conditions for Theorem 5.8 are satisfied.

Assumption (2) and Theorem 4.1 imply assumption (1). Assumption (3) and Proposition 4.5 imply assumption (1). □

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